Open Spin Chain and Open Spinning String

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Abstract

In this letter, we study the open spinning strings and their SYM duals. A new class of folded open spinning strings is found. At planar one-loop level in SYM, by solving the thermodynamic limit of the Bethe ansatz equations for an integrable open spin chain, we find good agreement with string theory predictions for energies of both circular and folded two-spin solutions. A universal relation between the open and closed spinning strings is verified in the spin chain approach.

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I. INTRODUCTION

Recently the integrable spin chains have become a powerful tool to study string/gauge theory duality [1]. On one hand, two special limits of IIB string theory in $AdS_5 \times S^5$ have been identified, the plane-wave limit [2, 3] and the semi-classical limit [4], in either of which the free closed string spectrum is known [5, 6]. On the other hand, on the dual gauge theory ($D = 4 \, \mathcal{N} = 4$ super Yang-Mills) side, to test these string predictions, one needs to compute the anomalous dimensions [7, 8] of corresponding gauge invariant composite operators. (For a recent review, see [9, 10] and references in.) In either of the above-mentioned limits, one needs to deal with mixing of a huge number of very long composite operators.

It was first observed in [11] that at least at the planar one-loop level, the mixing matrix of composite operators in a certain sector is equivalent to the Hamiltonian of an integrable spin chain. Thus the difficult problem of diagonalizing the anomalous dimension matrix of a huge number of long composite operators is reduced to solving the spin chain in a systematic manner with the algebraic Bethe Ansatz equations (ABAE) [12]. The great advantage of this approach is that by considering the thermodynamic limit of the Bethe ansatz, in the spirit of ref. [13], it can easily go beyond the BMN regime and be extended to the cases with many "impurities", such as the semi-classical spinning strings [14, 15]. Also it is accessible by numerical methods for both plane-wave strings and spinning strings as well [16, 17]. (For integrable spin chains in certain sectors of orbifold gauge theories with less supersymmetries ($\mathcal{N} = 1, 2$), see [18].)

Theory with both open and closed strings is known to have richer physics. The plane-wave/SYM duality has also been generalized to incorporate open strings [19, 20]. It is shown recently [21] that both Neumann and Dirichlet boundary conditions for open strings correspond, in the dual gauge theory, to integrable boundary terms in the pertinent open spin chain Hamiltonian. (In [22], the open integrable system with boundary has been identified in another context.) So the ABAE approach, together with whole open spin chain machinery (see, e.g., [23, 24, 25, 26]), can also be applied to the gauge theory that is dual to an open+closed string theory. This result is of principal interests, in that it extends the integrability to open strings and their gauge duals as well. Thus the appearance of integrable spin chain seems not accidental at all. To provide more evidence for the last statement, in this letter we use the integrable open spin chain approach to study the newly found open

spinning strings [27], and compare the results from the ABAE with the string predictions. As will be seen, once again we find good agreements. In this way, we add more evidence to the belief that there must be a profound relationship between string theory and spin chains.

In Sec. 2 we present folded open spinning strings, including a new class of solutions that rotate both in AdS_5 and S^5. Their energies is reproduced at planar one-loop level in SYM by solving the ABAE for an integrable open spin chain. In Sec. 4 we use the same approach to study the SYM dual of circular open spinning strings and of its fluctuations and get good agreement too. In both cases, an universal relation between closed and open spinning strings is verified in the spin chain approach. Sec. 5 is devoted to a short summary.

II. FOLDED OPEN SPINNING STRINGS

Here we follow the notations in [10], and present some open spinning string solutions that are relevant to our AdS/CFT test in the subsequent section. We will consider strings in $AdS_5 \times S^5/Z_2$ orientifold, whose dual is a conformal $\mathcal{N} = 2$ $Sp(N/2)$ gauge theory with matter [19]. For simplicity, we will suppress Chan-Paton factors, which are not essential to our discussions.

The bosonic part of the action for a string in $AdS_5 \times S^5$ reads

$$I = \frac{\sqrt{\lambda}}{2\pi} \int d\tau d\sigma (L_{AdS} + L_S) \tag{1}$$

where

$$
\begin{aligned}
L_{AdS} &= -\frac{1}{2}\eta^{PQ}\partial_a Y_P \partial^a Y_Q + \frac{1}{2}\tilde{\Lambda}(\eta^{PQ}Y_P Y_Q + 1), && P, Q = 0, \cdots, 5, \\
L_S &= -\frac{1}{2}g^{MN}\partial_a X_M \partial^a X_N + \frac{1}{2}\Lambda(g^{MN}X_M X_N - 1), && M, N = 1, \cdots, 6,
\end{aligned} \tag{2}
$$

with the metric $g^{MN} = \delta^{MN}$ and $\eta^{PQ} = \text{diag}\{-1, 1, 1, 1, 1, -1\}$. Here Λ, $\tilde{\Lambda}$ are the Lagrange multipliers imposing the constraints, respectively,

$$\eta^{PQ}Y_P Y_Q = -1, \qquad g^{MN}X_M X_N = 1 \tag{3}$$

As usual, for a closed string, the range of σ is from 0 to 2π, and X_M, Y_P satisfy periodic boundary conditions. For an open string, σ ranges from 0 to π and X_M, Y_P satisfy the Neumann or Dirichlet boundary conditions. It is often convenient to use the global

coordinates:

$$
\begin{aligned}
Y_1 + iY_2 &= \sinh\rho\sin\theta e^{i\phi_1}, & Z' = X_1 + iX_2 &= \sin\gamma\cos\psi e^{i\varphi_1} \\
Y_3 + iY_4 &= \sinh\rho\cos\theta e^{i\phi_2}, & W = X_3 + iX_4 &= \sin\gamma\sin\psi e^{i\varphi_2} \\
Y_5 + iY_0 &= \cosh\rho e^{i\phi_3}, & Z = X_5 + iX_6 &= \cos\gamma e^{i\varphi_3}
\end{aligned}
\tag{4}
$$

There are two classes of classical spinning string solutions: circular and folded. We first consider the folded ones, since their treatment in the spin-chain approach is easier. The folded closed two-spin classical solutions have been constructed in [6], and applied to test AdS/CFT far beyond the BPS regime [16]. There are two types of two-spin strings: those rotating on both AdS_5 and S^5, denoted as (S, J), and those rotating only on S^5, denoted as (J_1, J_2). They are actually related to each other by analytical continuation. On the gauge theory side, the corresponding gauge invariant operators are, respectively,

$$
Tr\mathcal{D}^S Z^J + \cdots, \qquad TrZ^{J_1}\Phi^{J_2} + \cdots . \tag{5}
$$

For an open string, one can construct similar classical solutions. For the (S, J)-type, one has

$$
\begin{aligned}
\theta &= 0, & \rho &= \rho(\sigma), & \phi_3 &= k\tau, \\
\gamma &= 0, & \phi_2 &= \omega\tau, & \varphi_3 &= \nu\tau
\end{aligned}
\tag{6}
$$

with ρ satisfying the Neumann boundary condition $\rho'|_{\sigma=0,\pi} = 0$.

From the equation of motion and Virasoro constraints, we have

$$
\begin{aligned}
\rho'' &= \sinh\rho\cosh\rho(k^2 - \omega^2) \\
\rho'^2 &= k^2\cosh^2\rho - \omega^2\sinh^2\rho - \nu^2
\end{aligned}
\tag{7}
$$

In the closed string one-fold case, σ ranges from 0 to 2π and could be divided into four segments: $\rho(\sigma)$ increases from 0 to ρ_0 when σ increases from 0 (or π) to $\pi/2$ (or $3\pi/2$), and then decreases back to 0 when σ goes on to π (or 2π). Here ρ_0 is the maximal value of ρ, determined by

$$
k^2\cosh^2\rho_0 - \omega^2\sinh^2\rho_0 - \nu^2 = 0. \tag{8}
$$

Therefore the turning point is at $\sigma = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, and $\sigma = 0, \pi$ are two singular points (fold-point). However, one has the freedom to translate the above points, keeping the relative

distance unchanged. So we can move the turning points to $\sigma = 0, \pi$, where $\rho' = 0$, and the singular point to $\sigma = \frac{\pi}{2}$. This is just what we need for a folded open string to satisfy the Neumann boundary condition. In this case, σ ranges from 0 to π and includes two segments. The energy, spin and angular momentum are

$$
\begin{aligned}
E &= 2\frac{\sqrt{\lambda}}{2\pi}k \int_0^{\rho_0} d\rho \frac{\cosh^2 \rho}{\sqrt{k^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho - \nu^2}} \\
S &= 2\frac{\sqrt{\lambda}}{2\pi}\omega \int_0^{\rho_0} d\rho \frac{\sinh^2 \rho}{\sqrt{k^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho - \nu^2}} \\
J &= 2\frac{\sqrt{\lambda}}{2\pi}\nu \int_0^{\rho_0} d\rho \frac{1}{\sqrt{k^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho - \nu^2}}.
\end{aligned}
\tag{9}
$$

The factor of 2 indicates that we have two segments of the string stretching from 0 to ρ_0. This class of folded spinning string solutions are a new result of our paper. Compared with the closed string case, we have

$$
E_o = \frac{1}{2}E_c(2S, 2J).
\tag{10}
$$

The dual operator in the gauge theory is proportional to

$$
Q\Omega\mathcal{D}^S(Z\Omega)^J Q + \cdots.
\tag{11}
$$

For the (J_1, J_2)-type solution [27], we have the ansatz

$$
\begin{aligned}
\rho = 0, \quad & \phi_3 = k\tau, \quad \gamma = \frac{\pi}{2} \\
\psi = \psi(\sigma), \quad & \varphi_1 = \omega_1\tau, \quad \varphi_2 = \omega_2\tau.
\end{aligned}
\tag{12}
$$

The equations of motion and Virasoro constraints are

$$
\begin{aligned}
\psi'' + \frac{1}{2}\omega_{21}^2 \sin 2\psi &= 0 \\
\psi'^2 + \omega_1^2 \cos^2 \psi + \omega_2^2 \sin^2 \psi &= k^2
\end{aligned}
\tag{13}
$$

where we have assumed without losing generality that $\omega_{21}^2 = \omega_2^2 - \omega_1^2 > 0$. In the same spirit as in above, one can put the turning point at $\sigma = 0, \pi$ and fold-point at $\sigma = \frac{\pi}{2}$. The maximal value of ψ_0 is determined by

$$
\omega_1^2 \cos^2 \psi_0 + \omega_2^2 \sin^2 \psi_0 = k^2.
\tag{14}
$$

Also the relation between energy and angular momenta can be read off from the closed string result:

$$E_o(J_1, J_2) = \frac{1}{2} E_c(2J_1, 2J_2). \tag{15}$$

The dual operators is of the form

$$Q\Omega(Z\Omega)^{J_1}(Z'\Omega)^{J_2}Q + \cdots . \tag{16}$$

In the following, we will show that on the gauge theory side, the anomalous dimensions of operators (16) do respect the relation (15) between energy and angular momenta, verifying the AdS/CFT correspondence for folded two-spin open strings.

III.   SYM DUAL FROM INTEGRABLE OPEN SPIN CHAIN

It was noticed in [21] that an integrable structure exists in the $\mathcal{N} = 2$ $Sp(N/2)$ gauge theory, at least at planar 1-loop level after turning off string interactions. More precisely, the anomalous dimension matrix for the open string BMN operators in the holomorphic scalar sector can be identified with the Hamiltonian of an $SU(3)$ open spin chain with integrable boundary terms. This observation provides a framework for dealing with one-loop mixing of a huge number of operators in SYM. We expect to recover the spectrum of classical spinning string solution (15) by solving the integrable spin chain.

For an integrable $SU(n)$ open spin chain, the Hamiltonian is given by [23, 26]

$$H_{open} = \sum_{m=1}^{L-1} H_{m,m+1} + \frac{1}{2\xi_-} \left. \frac{d}{du} K_{1,(l)}^-(u, \xi_-) \right|_{u=0} + \frac{\text{tr}_0[K_{0,(l)}^+(0, \xi_+) H_{L,0}]}{\text{tr} K_{(l)}^+(0, \xi_+)}, \tag{17}$$

where $H_{m,m+1} = I_{m,m+1} - P_{m,m+1}$ with $P_{m,m+1}$ the permutation operator. $K_{i,(l)}$ denotes the K-matrix acting on the Hilbert space $\mathcal{H}_i$ associated with the i-th site ($\mathcal{H}_0$ being the auxiliary space). The general diagonal K-matrix ensuring integrability has been obtained in [25]:

$$K_{(l)}^\pm(u, \xi_\pm) = \text{diag}\{\overbrace{a^\pm, ..., a^\pm}^{l}, \overbrace{b^\pm, ..., b^\pm}^{n-l}\}, \tag{18}$$

where

$$a^+ = i(\xi_+ - n) - u, \qquad b^+ = i\xi_+ + u,$$
$$a^- = i\xi_- + u, \qquad b^- = i\xi_- - u, \tag{19}$$

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with arbitrary $\xi_\pm$ and any $l \in \{1, ..., n-1\}$. The one-loop anomalous dimensions of open BMN operators consisting of holomorphic scalars (or eigenvalues of the Hamiltonian (17))[1] are given by [25]

$$\gamma = \frac{\lambda}{4\pi^2} \sum_{j=1}^{n_1} \epsilon(\mu_{1,j}), \qquad \epsilon(\mu) = \frac{4}{\mu^2 + 1}.$$ (20)

Here $\mu_{i,j}$ (the Bethe roots) satisfy the ABAE

$$\begin{aligned}
1 = {} & [e_{l-2\xi_-}(\mu_{l,k})e_{2\xi_+-l}(\mu_{l,k})\delta_{l,q} + (1 - \delta_{l,q})] \prod_{j=1}^{M_{q-1}} e_{-1}(\mu_{q,k} - \mu_{q-1,j})e_{-1}(\mu_{q,k} + \mu_{q-1,j}) \\
& \times \prod_{\substack{j=1, \\ j\neq k}}^{M_q} e_2(\mu_{q,k} - \mu_{q,j})e_2(\mu_{q,k} + \mu_{q,j}) \prod_{j=1}^{M_{q+1}} e_{-1}(\mu_{q,k} - \mu_{q+1,j})e_{-1}(\mu_{q,k} + \mu_{q+1,j})
\end{aligned}$$ (21)

for $k = 1, \cdots, M_q$ and $q = 1, \cdots, n-1$,

with $M_0 = L$, $M_n = 0$, $\mu_{0,j} = \mu_{n,j} = 0$, and

$$e_n(\mu) = \frac{\mu + in}{\mu - in}.$$ (22)

For the SYM dual of folded open two-spin solutions, two boundary terms in the Hamiltonian (17) should be of the form $\Sigma_1 = \Sigma(\otimes I_{3\times3})^{L-1}$, $\Sigma_L = (I_{3\times3}\otimes)^{L-1}\Sigma$, with $\Sigma = \text{diag}\{0, 0, 1\}$. The nonzero element in Σ corresponds to the Dirichlet boundary condition [19], while the zero diagonal elements in Σ correspond to the Neumann boundary condition. These requirements are satisfied by taking $\xi_+ = \xi_- = 1$ after setting $n = 3$, $l = 2$ in the K-matrix (18). The boundary parameters $\xi_\pm$ break the bulk $SU(3)$ symmetry of the spin chain down to $SU(2) \times U(1)$, the same as the R-symmetry of the gauge theory at hand.

For a folded open string of type (J_1, J_2), the dual operator (16) consists of J_2 Z's inserted in J_1 Zs. Or in open spin chain language, it consists of $L = J_1 + J_2$ sites and J_2 $\mu_{1,j}$-impurities, which are determined by the ABAE

$$\left(\frac{\mu_j + i}{\mu_j - i}\right)^{2L} \prod_{\substack{k=1 \\ k\neq j}}^{J_2} \frac{\mu_j - \mu_k + 2i}{\mu_j - \mu_k - 2i} \cdot \frac{\mu_j + \mu_k + 2i}{\mu_j + \mu_k - 2i}.$$ (23)

Here we have used $\mu_j = \mu_{1,j}$ to simplify the notation. We observe that these equations are invariant under $\mu_j \to -\mu_j$ for any single j. This manifests the fact that for an open

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[1] The expression of eigen-energies and the ABAE depend on the choice of the pseudo-vacuum ω. Here we choose $\omega = (\vec{v}\otimes)^{L-1}\vec{v}$ with $\vec{v} = (1, 0, ..., 0)^T$.

chain μ_j labels a standing wave, so the sign of μ_j is insignificant. This motivates us to use the following trick to solve these equations. Recall that the ABAE for a state of the corresponding closed spin chain, labelled as $(2L, 2J_2)$, with $2L$ the length of the spin chain and $2J_2$ the number of the impurities, is of the form

$$\left(\frac{\nu_j + i/2}{\nu_j - i/2}\right)^{2L} \prod_{\substack{k=1 \\ k \neq j}}^{2J_2} \frac{\nu_j - \nu_k + i}{\nu_j - \nu_k - i} . \tag{24}$$

When one requires that the Bethe roots distribute symmetrically, which is the simplest assumption to satisfy the trace condition, the above equations reduce to eq. (23). In other words, a state of the open spin chain, labelled by (L, J_2), is related to the state of a closed spin labelled by $(2L, 2J_2)$ with symmetrically distributed Bethe roots[2].

Since μ_j is of order L for large L, one can rescale μ's by $\mu_j = 4Lx_j$ and take the logarithm of the ABAE (23). One has

$$\frac{1}{x_j} = 2\pi n_j + \frac{1}{L} \sum_{\substack{k=1 \\ k \neq j}}^{J_2} \left(\frac{1}{x_j - x_k} + \frac{1}{x_j + x_k}\right). \tag{25}$$

The above equation is the same as the case for a closed chain state $(2L, 2J_2)$, Eq.(2.7) in [14], with symmetrically distributed Bethe roots. Thus in the spin chain approach we have

$$E_o = \frac{1}{2} E_c(2J_1, 2J_2). \tag{26}$$

This relation is the same as that between the folded two-spin open and closed strings suggested in ref. [27]. In this way, we have shown that the open spin chain in the gauge theory at planar one-loop level reproduces the energies of a folded open string. The (S, J)-type open spinning string we found in Sec. 2 should be described by an $SU(2, 2|2)$ super open spin chain, for which the relation (26) is expected to be valid too.

What is essential to our above treatment is the following observation for known folded (J_1, J_2)-type closed string solutions: In the "thermodynamic limit" $L \to \infty$, the Bethe roots in the complex plane are located along two cuts, which are almost parallel to the imaginary axis and symmetrical under reflection about the imaginary axis [14]. In the above we have been able to show that in the spin chain approach, in the limit $L \to \infty$, the Bethe roots on one of the cuts can be identified as the Bethe roots associated with an open spin chain state.

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[2] For large L one can ignore the minor difference between the two ABAE's caused by the root $\nu_k = -\nu_j$.

So the relation (26) follows naturally. Since the folded (S, J)-type closed spinning string is known to correspond to a distribution of Bethe roots along two cuts located on the real axis symmetrically about $\mu = 0$ [16], we expect the above trick should also work too.

IV. CIRCULAR OPEN SPINNING STRING AND OPEN SPIN CHAIN

Let us consider the circular two-spin open string solutions on $R_t \times S^5$, which have been studied in [27], with the ansatz

$$\rho = 0, \qquad \phi_3 = k\tau, \qquad \gamma = \gamma_0,$$
$$\psi = m\sigma, \quad \varphi_1 = \varphi_2 = \omega\tau, \quad \varphi_3 = \nu\tau, \tag{27}$$

where

$$k^2 = \nu^2 + (\omega^2 + m^2 - \nu^2)\sin^2\gamma_0,$$
$$\omega^2 = m^2 + \nu^2. \tag{28}$$

The corresponding energy and spins are

$$E = \frac{\sqrt{\lambda}}{2}k, \qquad J_3 = \frac{\sqrt{\lambda}}{2}\nu[1 - \frac{1}{2m^2}(k^2 - \nu^2)],$$
$$J_1 = J_2 = \frac{\sqrt{\lambda}}{8m}\sqrt{1 + (\frac{\nu}{m})^2(k^2 - \nu^2)}. \tag{29}$$

The open BMN string case corresponds to $\gamma_0 = 0$, so that $k = \nu$, $J_1 = J_2 = 0$ and $E = J_3 = J$. The characteristic frequency of the fluctuations around the classical solution, $\omega_n = \pm\nu \pm \sqrt{n^2 + \nu^2}$, leads to (for lower-energy modes)

$$\Delta E_o = -1 + \sqrt{1 + \frac{\lambda n^2}{4J^2}} = \Delta E_c(2J). \tag{30}$$

This is exactly the open string excited BMN spectrum in the plane-wave background [19], which can be recovered in dual gauge theory at planar one-loop level from an integrable $SU(3)$ open spin chain, as shown in [21].

Another special case is $\gamma_0 = \frac{\pi}{2}$, which leads to $J_1 = J_2 = J$, $J_3 = 0$ and

$$E = L\sqrt{1 + \frac{m^2\lambda}{4L^2}} = \frac{1}{2}E_c(2J_1, 2J_2), \tag{31}$$

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where $L = J_1 + J_2 = 2J$ and E_c is the energy of corresponding circular closed string. In this case, Z' directions satisfy the Neumann boundary conditions while W the Dirichlet conditions. The dual SYM operator is of the form

$$Q\Omega(Z'\Omega W\Omega)^J Q + \cdots. \tag{32}$$

Moreover, the characteristic frequency of the fluctuations around the classical solution (32),

$$\omega_n^2 = n^2 + 2\nu^2 + 2m^2 \pm 2\sqrt{(\nu^2 + m^2)^2 + n^2(\nu^2 + 2m^2)},$$

leads to spectrum of fluctuations as follows

$$\Delta E_o = \frac{\lambda}{8L^2} n\sqrt{n^2 - 4m^2} = \Delta E_c(2J_1, 2J_2). \tag{33}$$

Again, in the SYM dual, we can use an integrable $SU(3)$ open spin chain to calculate the anomalous dimension of the operators (32). Relative to the pseudo-vacuum that we have taken before, there are $L = 2J$ $\mu_{1,j}$-impurities and J $\mu_{2,j}$-impurities in the ABAE. Then in the thermodynamic limit, there should be two density functions for the Bethe roots, one for each class of impurities [13], which will leads to two coupled integral equations. To avoid this complication, we note that the operators (32) consist of Z' and W only, so one can restrict to an $SU(2)$ open spin chain, with the boundary terms $\Sigma_1 = \Sigma(\otimes I_{2\times2})^{L-1}$, $\Sigma_L = (I_{2\times2}\otimes)^{L-1}\Sigma$, $\Sigma = \mathrm{diag}\{0,1\}$. Thus, we are allowed to take the pseudo-vacuum to correspond to the operator consisting of Z' only. The above boundary terms are obtained by taking $n = 2$, $l = 1$ and $\xi_+ = 0$, $\xi_- = 1$ in the Hamiltonian (17). Then the ABAE (21) reads

$$\left(\frac{\mu_j + i}{\mu_j - i}\right)^{2(L+1)} = \prod_{\substack{k=1 \\ k\neq j}}^{J} \frac{\mu_j - \mu_k + 2i}{\mu_j - \mu_k - 2i} \frac{\mu_j + \mu_k + 2i}{\mu_j + \mu_k - 2i}. \tag{34}$$

It has been noted that the lowest energy eigenstates (with a fixed impurity number) of the Hamiltonian (17) with a diagonal K-matrix are also eigenstates of the shift operator $\hat{t} = \hat{P}_{1,2}\hat{P}_{2,3}\cdots\hat{P}_{L-1,L}$ [25], where $\hat{P}$ is the permutation operator. Moreover, it is easy to check that the operator (32) is unchanged if $\hat{t}$ acts on it twice. This observation yields, in the special case with $L = 2J$, the constraint

$$\prod_{j=1}^{J} \frac{\mu_j + i}{\mu_j - i} = \pm 1. \tag{35}$$

This constraint looks similar to the trace condition for a closed chain, and implies the Bethe roots distribute symmetrically about the origin.

The same way as the closed string/closed chain correspondence, we expect that the imaginary Bethe roots in an open chain correspond to a circular open string state. For simplicity, we consider the case with J odd. We may rescale the Bethe roots by $\mu_j = 2iq_jL$ with q_j real, and take logarithm of the above equations. For large L they become

$$\frac{1}{q_j} = \frac{2}{L} \sum_{\substack{k=-\frac{J-1}{2} \\ k \neq j}}^{\frac{J-1}{2}} \frac{1}{q_j - q_k} \quad \xrightarrow{k/J \to x} \quad \int dq \frac{\sigma(q)}{q_j - q}, \tag{36}$$

where the integral is understood as the principal value, and $\sigma(q)$ is the root density.

The important observation in [14] is that the imaginary Bethe roots μ_j for a closed spinning chain form a "condensate" in the interior of an interval of order of L along the imaginary axis around the origin. For the scaled variable q_j, the condensate region is of order of unity. Outside the condensate region, the Bethe roots start to spread out. We expect a similar situation happens to the open chain as well. Hence we make the following ansatz for the root density:

$$\sigma(q) = \begin{cases} 2m & -s < q < s, \ m = 1, 2, ..., \\ \tilde{\sigma}(q) & s < q, \\ \tilde{\sigma}(-q) & q < -s. \end{cases} \tag{37}$$

Note our ansatz here is more general than that in [14]. There are several remarks related to this ansatz:

- From the root density, it is easy to see that in the condensed region, one has $-i\mu_j \simeq 2j/m$, with integer $j \in (-msL, \ msL)$.

- Part of equations in the ABAE (34) can be reduced to

$$\left(\frac{2j+m}{2j-m}\right)^{(L+1)} = \prod_{\substack{k=-msL \\ k \neq j}}^{msL} \frac{j-k+m}{j-k-m} \times [1 + O(\frac{1}{L})]. \tag{38}$$

  For finite j the left hand side of the above equations, in the limit $L \to \infty$ increases (or decreases) exponentially; it has to be compensated by a small denominator on the right hand side. Therefore, m has to be an integer.

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- The $m = 2$ case is the same as for a closed string (or a closed chain) considered in [14]. There the roots $\mu \to \pm i$ have to be included in order to ensure the trace condition if the root $\mu_J = 0$ is introduced. It implies that m has to be even[3].

- It is straightforward to check that the constraint (35) is satisfied with this ansatz, for both of m even and m odd. The crucial point is that $\pm i$ are not roots for m odd, then $\mu_J = 0$ contributes a factor of -1 to the left hand side of (35).

With the ansatz (37) we can directly follow the method used in sect. 4.1 of [14], and obtain the solution of the integral equation (36):

$$\tilde{\sigma}(q) = \frac{1}{\pi}\sqrt{q^2 - s^2}\left(-\frac{1}{qs} + 2m\int_{-s}^{s}\frac{dv}{(q-v)\sqrt{s^2 - v^2}}\right),\tag{39}$$

with the consistency condition

$$\int_{-s}^{s}\frac{dv}{\sqrt{s^2 - v^2}} = \frac{1}{2ms}.\tag{40}$$

The above consistency condition yields $s = 1/2m\pi$. Substituting Eq. (40) into (39), one can directly check that the solution satisfies the normalization condition $\int dq\sigma(q) = 1$. The anomalous dimension is found, by using (20), (37) and (39), to be

$$\gamma = -\frac{\lambda}{8\pi^2}\left(\sum_{j=-msL}^{msL}\frac{4m^2}{(2j)^2 - m^2} + \frac{1}{L}\int_{s}^{\infty}dv\frac{\tilde{\sigma}(v)}{v^2}\right)$$

$$= \frac{m^2\lambda}{8L}.\tag{41}$$

This result precisely matches the spectrum for a circular solution (31) when $\lambda/L \ll 1$. As mentioned above, m could be any integer, even or odd, in the open chain case.

Another way to solve this (L, J) open chain system is to compare the ABAE (34) with the ones for a $(2(L+1), 2J)$ closed chain system with Bethe roots distributing symmetrically about $\mu = 0$. In [14], the state with an odd number of Bethe roots has been studied thoroughly. One expects that the thermodynamic limit is independent of whether the number of Bethe roots is odd or even and, therefore, $E_o = \frac{1}{2}E_c(2L, 2J)$ holds true generally.

Finally let us consider the fluctuations around the ground state (41). It corresponds to adding a few new roots or move a few roots away from the imaginary axis. One lesson from

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[3] Note that it is $m/2$ that is the winding number of the closed circular string.

the spin chain description of the BMN spectrum is that the fluctuations correspond to a set of Bethe roots with a few of them moving onto the real axis. The spinless fluctuations, that is to move a few roots from the imaginary axis to the real axis, do not change the filling factor $\alpha = J/L = 1/2$. To maintain the constraint (35), the roots should distribute symmetrically about the origin. In other words, the roots are moved in pairs and located on the real axis symmetrically. Let us consider a pair of roots at $\pm 2\mu L$ with μ real and positive. In the large L limit, μ satisfies the equations

$$\frac{1}{\mu} = n\pi + \int dq \frac{\sigma(q)}{\mu - iq}. \tag{42}$$

Substituting the solution (39) for the root density into the above equation, we determine

$$\mu^{-1} = \pi\sqrt{n(n+4m)}, \tag{43}$$

which contributes to the anomalous dimensions

$$\gamma_\mu = \frac{2\lambda}{8\pi^2 L^2 \mu^2} = \frac{n(n+4m)\lambda}{4L^2}. \tag{44}$$

On the other hand, the roots at $\pm 2\mu L$ also back react on the roots on the imaginary axis . It shifts s to

$$s \simeq \frac{1}{2m\pi}\left(1 - \frac{4}{L}\frac{1}{\sqrt{1+(2m\pi\mu)^2}}\right), \tag{45}$$

and the anomalous dimension to

$$\gamma_{\text{ir}} = \frac{\lambda}{32\pi^2 L}\left(\frac{1}{s^2} - \frac{8}{L\mu^2}\left(1 - \frac{1}{\sqrt{1+(2m\pi\mu)^2}}\right)\right). \tag{46}$$

Substituting Eqs. (43) and (45) into (46) and adding (44), we find that the change in the anomalous dimension (41) is

$$\Delta\gamma = \frac{\lambda}{4L^2}(n+2m)\sqrt{n(n+4m)} = \frac{\lambda}{4L^2}n'\sqrt{n'^2 - 4m^2}. \tag{47}$$

This precisely doubles the spectrum (33) predicted by the fluctuations of open circular string. It just reflects the fact that two fluctuation modes have been counted.

If we add a few new roots on the real axis, the total filling factor[5] will become greater than 1/2. Therefore, the constraint (35) will be broken down. In other words, the new

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[5] The change in the filling factor of order $1/L$ will change the ground state energy by an amount of order $1/L^3$ only[14].

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roots at the real axis need not to be in pairs and be symmetrical. The simplest case is to consider a single new root added. The calculation, however, is much more subtle than that for spinless fluctuations. We do not present the details here, but merely mention we have verified that the change in the anomalous dimension (41) is indeed half of the one in (47), as expected.

V. CONCLUSION

In [21], we have identified an integrable structure in a conformal $\mathcal{N} = 2$ $Sp(N/2)$ gauge theory, at least at planar one-loop level. In the open-string holomorphic scalar sector, it is an integrable $SU(3)$ open spin chain. In this letter, we explore this structure to test the AdS/CFT duality for open spinning strings. It is shown that the spectrum of both a circular and folded open spinning string can be reproduced at planar one-loop level in SYM by examining the thermodynamic limit of the Bethe ansatz for the open spin chain, in the same spirit as in ref. [13]. We have verified that the classical energy of the open spinning string is always related to that of a closed string counterpart by

$$E_o = \frac{1}{2} E_c(2J_1, 2J_2, \cdots). \tag{48}$$

This relation was first found in three-spin solutions in [27]. Moreover, we also verified that fluctuation spectrum around classical circular open string states can be reproduced in SYM by means of the open spin chain. The fluctuation energy of open spinning string is related to that of a closed string counterpart by

$$\Delta E_o = \Delta E_c(2J_1, 2J_2, \cdots). \tag{49}$$

It is slight different from the relation between classical energy. From the open spin chain's point of view, both of two relations reflects the doubling trick, namely the length of the chain gets doubled and the impurities form mirror images, due to the existence of the ends of the chain. From the viewpoint of string theory, it just reflects the doubling trick in 2d conformal field theory with boundary. Roughly speaking, a closed string can be constructed from two copies of an open string. Thus, the validity of this relation in both approaches naturally suggests a more direct relation between a string and a spin chain; i.e. a spin chain may be viewed as a discrete version of a string. More examples to check this relation are welcome.

There exist three-spin open spinning solutions, as shown in [27]. It would be very interesting to study the SYM duals of such solutions in terms of the open spin chain.

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